International workshop
"Distribution of power and voting procedures in the European Union"
Natolin European Centre, Warsaw
October 12-13, 2007

ANDREA FRANCHI[1], GIANFRANCO GAMBARELLI[2], ANGELO URISTANI[3]

OLD AND NEW ELECTORAL SYSTEMS[4]

1. ONE-DISTRICT ELECTORAL SYSTEMS

1.1. GOALS

Consider a political system composed of three parties. These parties respectively gain 50, 30 and 20 votes in an election. Let's suppose there are five seats to assign. Table 1 shows the *Hare quotas,* that is, votes multiplied by the ratio "total seats / total votes", and all the possible assignments that comply with *monotonicity* criterion (if a party has more votes than another one, it does not have fewer seats). If governability is to be given priority, we should use the apportionment α (*majority system*) or β (*majority system respecting minorities*) or γ (20% *threshold*). If democraticity is to be given priority, we have to use an apportionment without threshold barriers, as close as possible to exact quotients, i.e. δ or ε. If respect of minorities has to be guaranteed, we can use δ or ε again.

1.2. CRITERIA

Apart from monotonicity, other equity criteria have been introduced as *equal seats for equal*

[1] Faculty of Economics, University of Bergamo, Italy, e-mail: andrea.franchi@studenti.unibg.it.

[2] Department of Mathematics, Statistics, Computer Science and Applications "Lorenzo Mascheroni", University of Bergamo, Italy, e-mail: gambarex@unibg.it, http://dinamico.unibg.it/dmsia/staff/gambar.html.

[3] University of Bergamo, PhD student in Computational Methods for Forecasting and Decisions in Economics and Finance, e-mail: angeluri@yahoo.it.

[4] Sponsored by MIUR. The authors thank Barbara Pezzini for useful suggestions.

votes and *symmetry* i.e. the apportionment of seats does not depend on the party's order considered.

Amongst the most popular criteria, there is the respect of *Hare:* the number of seats to assign to each party has to be lower than the respective rounded up Hare quota (*Hare maximum*); moreover, the number of seats has to be greater than the respective rounded down Hare quota(*Hare minimum*).

In our example the Hare minimum is respected only in the apportionments δ and ε; the Hare maximum only from γ, δ and ε; the apportionments that respect both Hare are δ and ε.

	votes	Hare quotas	seats				
			α	β	γ	δ	ε
A	50	2.5	5	4	3	3	2
B	30	1.5	0	1	2	1	2
C	20	1.0	0	0	0	1	1
tot.	100	5.0	5	5	5	5	5

Table 1. Apportionments of seats that comply with *monotonicity* criteria

Another significant criterion is *superadditivity*; which consists of the following. If the rounding method assigns some seats to two parties, the same method has to assign to the united party (obtained by a hypothetical coalition between the two parties) a number of seats not lower than the amount of seats previously assigned to each single parties.

A more recent criterion, related to parities' coalitional power, was introduced in (Gianfranco Gambarelli and Jerzy Holubiec, 1990). We briefly explain it here with reference to the previous example. On the basis of the votes obtained, no party that has less than 50% can be a majority. The only possible majorities are the coalitions (A, B), (A, C) and (A, B, C). A party is named *crucial* for a coalition if the last is a majority coalition which becomes a minority, if the party leaves it. In our case, party B is crucial for the coalition (A, B); party C is crucial for (A, C); party A is crucial for the three majority coalitions. The *normalized index of Banzhaf* (John Banzhaf, 1965) assigns to every party a power proportional to the number of coalitions for which it is crucial; in our case 1/5 to B and C, 3/5 to A. There are other indices for other applications, but in the electoral context the normalized index of Banzhaf is the most adapted, because of the grip proportionality; starting from now with the term *"index of power"* we will make reference to the one previously described. For further information we suggest to look (Gianfranco Gambarelli and Guillermo Owen, 2004).

As we have seen, the indices corresponding to the distribution of votes are (3/5, 1/5, 1/5). If now we consider the distribution of seats, we can easy verify that the index is worth 1 for A (majority alone) and 0 for the other two. The same occurs for β, γ and δ. The indices corresponding to the distribution ε are instead (1/3, 1/3, 1/3) because every party is crucial for two coalitions. According to the *criterion of the power index*, the maximum democracy is guaranteed by the first four distributions, because they are nearest (according to an interpretation explained later) to those of the votes, in terms of power indices.

1.3. CONTRADICTIONS

The application of these and other criteria all seem necessary, but they can bring contradictions. For example, if there are two parties and they receive the same number of votes, it is impossible to assign an uneven number of seats without breaking the criterion "equal seats for equal votes" and therefore the symmetry. At the same way, if the votes obtained by the three parties are (49, 49, 2) and there is a single seat to assign, the respect of the criterion "equal seats for equal votes" would assign the seat to the third party, with obvious infraction of the monotonicity. More generally, it has been proved that whichever method of rounding which safeguards symmetry and monotonicity, cannot jointly respect Hare maximum and superadditivity.

1.4. CLASSICAL ROUNDING METHODS

We will now give a short presentation of the rounding methods more commonly used. According to the *pure proportional system* (of Hamilton) each party is initially assigned the entire part of the Hare quotas (in the case of the example draft is (2, 1, 1)). The residual seats (in our case one) are given to the parties that have an higher fractional part of the exact quotients. In our example, both A and B have fractional part 0.5; so there are two possible distributions: δ and ε. The decision on which of the two will be adopted is taken by outside factors: age of the candidates, destiny and so on. Even in other methods, for equal conditions outside factors are used.

According to the *method of greatest divisors* (of Hondt) the procedure is the following. The votes gained by the first party are divided by 1, by 2, by 3 and so on, as long as the procedure requires it. Analogous divisions are computed for the votes gained by the other parties. The *S* highest quotients are considered (where *S* is the number of seats to assign) and a seat is attributed to each party corresponding to such quotients. In the case of our example (see Table 2) the higher quotients are, in decreasing order, 50, 30, 25, 20, $16,\bar{6}$; from these quotients three correspond to A, one to B and one to C. Therefore three seats are assigned to the first party and a seat to each of the other two. The resulting distribution is δ.

The method of greatest divisors can produce apportionments that do not respect the Hare maximum. For this purpose, Michel Balinski and Peyton Young introduced in (1975) the *method of greatest divisors with quota*, where the eventual surplus regarding the rounding for excess are not assigned to the interested party. As we will see later, such technique has other disadvantages.

Other variations are carried by the *method of greatest divisors with jump* (St. Lague, Niemeyer and so on). They consist in the divisions not by 1, 2, 3, 4... but by more distanced sequences, like 1, 3, 5, 7 etc; small parties benefit from these methods.

VOTES	:1	:2	:3	. . .
50	<u>50</u>	<u>25</u>	<u>16,</u> $\overline{6}$	. . .
30	<u>30</u>	15	10,0	. . .
20	<u>20</u>	10	6, $\overline{6}$	. . .

Table 2. Apportionment according to the *method of greatest divisors.*

1.5. THE METHOD OF MINIMAX

The classic rounding methods have this approach: "we apply a technique, then we see which criteria it violated, and we cry over it" (in the case of greatest divisors with quota only one violation is solved). *The method of minimax* (Gianfranco Gambarelli, 1999) overturns this approach by proceeding as follows. Taking into account that all the equity criteria cannot be respected totally (see § 1.3), the damage has to be limited as much as possible. Therefore, we base the list (in order of preference) on the criteria we think are the most important. We build the set of seat distributions that satisfy the first criterion. Then we isolate the subset of seats that also satisfy the second criterion (if not empty) and we proceed until we obtain a final set of apportionments. As we will see, this final set is not empty. If during the restrictions we reach a single apportionment, then we stop the calculation. If at the end several apportionments still remain, we proceed with external methods in order to obtain a unique solution.

The first criterion to use is the combined respect of the monotonicity, Hare minimum and Hare maximum. It is possible to prove that such criterion generates in each case a non empty set of apportionments.

As a second criterion, the *minimization of the maximum damage* is adopted, according to the procedure of David Schmeidler (1969) that we will explain later. This minimization is subsequently applied to the percentage apportionment of votes and to one of the power indices (or vice-versa, if the last one is considered prioritary). This procedure implies the respect of "equal seats for equal votes" and in general symmetry, in cases where this is possible (see Section 1.3). These operations reduce the possible set, without making it empty.

We return to the example illustrated in Table 1. Firstly, we eliminate all the apportionments not respecting the Hare (as we know, the Table already excludes those that do not respect monotonicity). The remaining distributions are δ and ε. We begin with the minimization of the maximum damage in terms of vote apportionments.

	votes %	δ%	damages
A	50	60	-10
B	30	20	10
C	20	20	0

ordered damages
10
0
-10

Table 3. The damage of δ in terms of votes.

	votes %	ε%	damages

ordered damages

A	50	40	10	10
B	30	40	-10	0
C	20	20	0	-10

Table 4. The damage of ε in terms of votes.

Table 3 shows the apportionment percentages of votes, the apportionment percentages of seats corresponding to δ, the percent differences between votes and seats (i.e. the damage caused by that distribution to each single party) and in the last column the damages reordered from the highest to the lowest. Observe that the damage for A is negative (it is an advantage) because the distribution δ supports this party. We repeat the same procedure with ε (see Table 4). Notice that the ordered damages are the same for both the apportionments, therefore an this stage we do not eliminate any of them and we continue to the next minimization.

We show in Table 5, in order, the power indices relative to the votes, those relative to distribution δ, the damages and the ordered damages. We do the same in Table 6 for distribution ε. At this point we compare the ordered damages. Observe that the maximum damage brought by the apportionment ε (= $26,\overline{6}$) is higher than the maximum damage brought by δ (= 20). Therefore we reject ε and obtain the final apportionment δ = (3, 1, 1).

The procedure of Schmeidler used here concerns a concept of solution for cooperative games: the *nucleulus*. It consists in choosing, from all possible solutions, the one(s) that minimizes the maximum damage; in case of parity, the first element of the damage sequences is neglected and the maximum, amongst the remaining damage, is minimized, and so on. In the previous stage (Tables 3 and 4) the procedure did not make a selection, because the order sequences of the damages were identical.

	p.i. of votes x 100	p.i. of δ x 100	damages
A	60	100	-40
B	20	0	20
C	20	0	20

ordered damages
20
20
-40

Table 5. The damages of δ in terms of power indices.

	p.i. of votes x 100	p.i. of ε x 100	damages
A	60	$33,\overline{3}$	$26,\overline{6}$
B	20	$33,\overline{3}$	$-13,\overline{3}$
C	20	$33,\overline{3}$	$-13,\overline{3}$

ordered damages
$26,\overline{6}$
$-13,\overline{3}$
$-13,\overline{3}$

Table 6. The damages of ε in terms of power indices.

1.6. INTERESTED PARTIES

It is possible to use mixed methods; for example the thresholds and/or the majority bonus can be applied together with the proportional system or with the greatest divisors.

We summarise the advantages of the various methods for several types of party.

For the party (or the coalition) that gained a relative majority, firstly the majority system is advantageous and secondly the majority bonus.

Large and medium-large parties have an advantage with thresholds.

For small parties with cultural and/or language peculiarities, the respect of minorities gives an advantage.

For small and small-medium parties, firstly the greatest divisors with quota and/or jump is advantageous, and secondly the proportional system.

No party has an advantage with the minimax method, because it respects all the main criteria of equity and minimizes distortions as much as possible.

2. MULTI-DISTRICT ELECTORAL SYSTEMS

2.1. THE PROBLEM BECOMES COMPLICATED

Consider a Parliament composed of 11 seats coming from two electoral districts (6 seats in the first one, 5 seats in the second one). Let us suppose there are only three parties: A, B and C. The votes gained after an election are shown in Table 7. We quote in Table 8 the Hare quotas at national level (i.e. relative to the total of the two districts) and in Table 9 the Hare quotas at local level.

VOTES	A	B	C	totals
District I	50	60	10	120
District II	10	10	60	80
National totals	60	70	70	200

Table 7. The votes gained.

National Hare quotas	3,30	3,85	3,85	11

Table 8. The Hare quotas at national level.

Local Hare quotas	A	B	C	totals
I	2,500	3,000	0,500	6
II	0,625	0,625	3,750	5
totals	3,125	3,625	4,250	11

Table 9. The Hare quotas at local level.

Notice that the Hare quotas at national level (Table 8) are different from the totals of the Hare quotas at local level (last line of Table 9). This involves the first disadvantage in terms of *symmetry of the totals*, because parties B and C have equal national totals (= 70), but they obtain different apportionments of seats. The distortion can become significant after the rounding, because some methods can break the *monotonicity of the totals,* i.e. more total seats with fewer total votes.

Table 10 shows the corresponding power indices. Tables 11, 12 and 13 show the seats corresponding to the application of the three main rounding methods in each electoral district.

p.i. of votes	A	B	C	totals
I	20	60	20	100
II	0	0	100	100

Table 10. The power indices related to the votes.

Hamilton (1)	A	B	C	totals
I	3	3	0	6
II	1	0	4	5
totals	4	3	4	11

Hamilton (3)	A	B	C	totals
I	3	3	0	6
II	0	1	4	5
totals	3	4	4	11

Hamilton (2)	A	B	C	totals
I	2	3	1	6
II	1	0	4	5
totals	3	3	5	11

Hamilton (4)	A	B	C	totals
I	2	3	1	6
II	0	1	4	5
totals	2	4	5	11

Table 11. The seats assigned by the proportional system (4 cases).

Hondt	A	B	C	totals
I	3	3	0	6
II	0	0	5	5
totals	3	3	5	11

Table 12. The seats assigned by the method of greatest divisors.

Bal.&Young (1)	A	B	C	totals
I	3	3	0	6
II	0	1	4	5
totals	3	4	4	11

Bal.&Young (2)	A	B	C	totals
I	3	3	0	6
II	1	0	4	5
totals	4	3	4	11

Table 13. The seats assigned by the greatest divisors with quota.

2.2. THE VIOLATIONS CAUSED BY CLASSICAL METHODS

It is easy to verify that the application of every above method involves at least one violation.

In particular:

- the proportional system, in the four cases shown in Table 11, does not respect the criteria of: power indices in the first electoral district (cases 1 and 3), symmetry of totals (cases 1, 2 and 4), monotonicity of totals (case 1) and Hare maximum of totals (cases 2 and 4);
- the method of greatest divisors violates the local criterion of the power indices (I district), the local criterion of Hare maximum (II district) and the criteria of the Hare maximum and of the symmetry of totals (see Table 12);
- the method of greatest divisors with quota does not respect the local criterion of power indices (I district) in both the possible configurations.

2.3. THE MINIMAX METHOD FOR MULTI-DISTRICT SYSTEMS

This method, introduced by Gianfranco Gambarelli and Arsen Palestini in 2007, is a generalization of the minimax method at one-district system. We start from an ordering of the criteria, as an example:

- Monotonicity, Hare maximum and minimum at national level;
- Minimization of the maximum damage in terms of votes percentages at national level;
- Minimization of the maximum damage in terms of power indices at national level;
- Monotonicity, Hare maximum and minimum at local level;
- Minimization of the maximum damage in terms of power indices at local level,
- Minimization of the maximum damage in terms of votes percentages at local level.

As for the one-district minimax method, we start by considering the possible distributions of seats that respect the first criterion, then we select the subset of the ones that also respect the second, and so on, till the last criterion. It is possible to prove that, if the first criterion is "Monotonicity, Hare maximum and minimum at national level", then the set of survival distributions is not empty.

The minimization of the maximum damage is achieved (both at national and at local level) using the minimax technique shown in Section 1.5. We illustrate it by referring to the previous example.

Consider a generic distribution of totals resulting from the application of the minimax method at national level. Build a distribution of seats amongst the districts, in order to respect in each monotonicity, Hare maximum and minimum (for example it could be the greatest divisors with quota). Compute the relative power indices and the differences from those of the votes (see Tables 13 and 14). Put these differences in a decreasing order. Repeat the procedure for all other possible seat apportionments that follow the above described process. Compare the ordered sequences of the differences and choose the distributions of seats corresponding to the sequences that have the first element minimum; on equal terms, that have the second one minimum and so on.

If the remaining distributions are more then one, we proceed in the same way minimizing the maximum damage in terms of vote percentages

In our case the procedure leads, already at level of power indices, to the only solution described in Table 15. Table 16 shows the calculations. Notice that the ordered sequence of damages in the case of the greatest divisors with quota (Table 13) is (20, 10, 0, 0, 0, -30) while it is void in the case of minimax solution.

	Power indices (x100) of seats				Differences p.i. votes-seats			
Districts	A	B	C	totals	A	B	C	totals
I	50	50	0	100	-30	10	20	0
II	0	0	100	100	0	0	0	0

Table 14. The power indices of seats (Table 13) and the differences from those of votes.

Minimax	A	B	C	totals
I	2	3	1	6
II	1	1	3	5
totals	3	4	4	11

Table 15. The seats assigned by the minimax method, using the ordering of criteria adopted here.

	Power indices (x100) of the seats				Difference p. i. votes-seats			
Elect.Dist	A	B	C	totals	A	B	C	totals
I	20	60	20	100	0	0	0	0
II	0	0	100	100	0	0	0	0

Table 16. The power indices of seats in Table 15 and the differences from those of the votes.

2.4. WHY THE MINIMAX SOLUTION RESPECTS THE CRITERIA

As we can see, the solution of our example respects, at local and at national level, symmetry, monotonicity, Hare minimum and maximum, equal seats for equal votes and power indices. As we have shown in Section 2.2, is not the same for any of the classical methods. This happens because the solution has been built specifically to respect these criteria. The paper (Gambarelli and Palestini, 2007) shows the mathematical formulation, a theorem of existence of the solution and a generating algorithm. Of course, the application of this method to real Parliaments requires the use of computers. Obviously the computer had not been invented when the classical methods were introduced. Now we have them and can use them to achieve better solutions.

2.5. INTERESTED PARTIES

Of course, the choice of the ordering of the criteria influences the type of solution. In particular, the main decision lies in the priorities at national or local level. In the first case, the large and medium parties on national territory have an advantage; in the second case there are benefits for minorities concentrated in particular geographical areas. What was said in Section 1.6 about the advantages/disadvantages of the various methods is also valid for multi-district systems.

As we have shown, the minimax method does not give any benefit to any party, because it respects all the main criteria of equity and minimizes the distortions. For these reasons, it will probably never be adopted.

APPENDIX: A Prescriptive formulation of the "minimax" method

We hereby explain a prescriptive formulation of the minimax method, based on a particular ranking of the criteria. We do not use direct techniques (such as the Hamilton method) so that the formulation may be easily modified in case of a different ranking. Consider also that an algorithm is available to calculate automatically the subdivision of seats using the minimax method (see Gambarelli and Palestini, 2007).

Art.1. Definitions concerning the votes

1. For the sharing out of seats the following definitions have to be used:

   a) the "value of majority concerning votes"of a group of parties is equivalent to 1, if the sum of valid votes of the related parties is above 50% of the sum of total valid votes; it is equivalent to 0 in all other cases;

   b) the "absolute cruciality concerning votes" of a party is equivalent to the number of the different groups of parties, each of which has value of majority concerning votes equivalent to 1 and, if lacking that party, has value of majority concerning votes equivalent to 0;

c) the "total cruciality concerning votes" is equivalent to the sum of all absolute crucialities concerning votes, of the parties of the considered electoral system;

d) the "relative cruciality concerning votes" of a party is equivalent to the ratio of absolute cruciality concerning votes, of that party, and the total cruciality concerning votes;

e) the "votes quotient" of a party is equivalent to the ratio of the number of valid votes of that party and the number of total valid votes;

Art.2. Definitions concerning the seats

1. For the sharing out of seats the following definitions are used:

a) the "value of majority concerning seats" of a group of parties is equivalent to 1, if the sum of the seats assigned to those parties is above 50% of the total seats; it is equivalent to 0 in all other cases;

b) the "absolute cruciality concerning seats" of a party is equivalent to the number of the different groups of parties each of which has "value of majority concerning seats" equivalent to 1 and, if lacking that party, has value of majority concerning seats equivalent to 0;

c) the "total cruciality concerning seats" is equivalent to the sum of all absolute crucialities concerning seats, of the parties of the considered electoral system;

d) the "relative cruciality concerning seats" of a party is equivalent to the ratio of absolute cruciality concerning seats, of that party, and the total cruciality concerning seats;

e) the "quotient of seats" of a party is equivalent to the ratio of the number of seats assigned to that party and the total number of seats provided.

Art. 3. Definitions concerning the connection between votes and seats

1. For the sharing out of seats the following definitions are used:

a) the “number of theoretical seats” of a party is equivalent to the quotient of votes of that party multiplied by the total number of seats to be assigned;

b) the “lowest limit” of a party is equivalent to the number of its theoretical seats rounded down;

c) the “highest limit” of a party is equivalent to the number of its theoretical seats rounded up, if not integer; it is equivalent to the number of its theoretical seats, if integer;

d) the “difference of percentage between votes and seats” of a party is equivalent to the difference between its quotient of votes and its quotient of seats;

e) the “difference of crucialities between votes and seats” of a party is equivalent to the difference between its relative cruciality concerning votes and its relative cruciality concerning seats.

Art. 4. Procedures to distribute seats in one constituency cases

1. The electoral office may use, when executing procedures for distributing seats, computer tools, whose structure, physical and logical, has previously been authorized by the Ministry.
2. The electoral office identifies the lists of all possible divisions among all parties, of the seats to be assigned and from that list it selects progressively sub categories of possible divisions as specified in the following subsection 3. The procedure is complete when the sub category identified at the end is composed of a sole division of seats and that represents the final division.
3. The electoral office proceeds according to the following rules:

   a) it selects the list of all possible divisions so that the number of seats assigned to each party is not below its lowest limit and not above its highest limit;

b) among the lists of all possible divisions above mentioned in section 3.a, it selects the lists of divisions that attribute, to each party that has received more votes than another party, a number of seats not lower than the number of seats assigned to the other party;

c) among the lists of divisions mentioned in section 3 b, it selects those lists for which the highest difference of percentage votes-seats is the lowest difference; if the resultant list is made up of more than one division, it selects from that list those divisions in which the highest difference of percentage votes-seats, except that which has been counted, is the lowest difference, and it proceeds in the same way until the last difference has been calculated;

d) among the lists of divisions mentioned in section 3 c, it selects those lists for which the highest difference of crucialities is the lowest; if the resultant list is made of more than one division, it selects from that list those divisions in which the highest difference of crucialities, except that which has been counted, is the lowest difference, and it proceeds in the same way until the last difference of crucialities has been calculated;

e) among the lists of divisions mentioned in section 3.d, it selects [... *at this point further selections, based on suitable exogenous criterions (sex or age of candidates, draws by lot etc.) have to be illustrated in order to identify a sole final division*].

Art. 5. Procedures to distribute seats in more than one constituency cases

1. The electoral office may use, when executing procedures for distributing seats, computer tools, whose structure, physical and logical, has previously been authorized by the Ministry.
2. The electoral office identifies the lists of all possible divisions of the total seats among parties, obtained using the procedure described in article 4, with the exception of section 4.e. It then identifies the lists of the local divisions of the seats, which respect both the total seats

that have to be assigned in each constituency, and any division of the total seats among he parties, (division) obtained as described in the previous article.

3. Among the lists of local divisions, as described in section 5.2, it (the electoral office) selects progressively sub categories of local divisions, according to the procedure described in the following section 5.2. The procedure is complete when the sub category in that manner selected is made up of a sole local division of seats and that represents the final division.
4. The electoral office proceeds according to the following rules:

   a) among the lists of the divisions mentioned in secton 5.1, it selects those lists for which the highest difference of percentage votes-seats of all the parties in all constituencies is the lowest difference; if the resultant list is made up of more than one division, it selects from that list those divisions in which the highest difference of percentage between votes-seats, except that which has been counted, is the lowest difference, and it proceeds in the same way until the last difference of percentage between votes-seats has been calculated;

   b) among the lists of divisions mentioned in section 5.2.a, it selects those lists for which the highest difference of crucialities of all parties is the lowest; if the resultant list is made up of more than one division, it selects from that list those divisions in which the highest difference of crucialities, except that one which has been counted, is the lowest difference and it proceeds in the same way until the last difference of crucialities has been calculated;

   c) among the lists of divisions mentioned in section 5.2.b it selects [... *at this point further selections, based on suitable exogenous criterions (sex or age of candidates, draws by lot etc.) have to be illustrated in order to identify a sole final division*].

REFERENCES

M. BALINSKI, H. PEYTON YOUNG, *The Quota Method of Apportionment*, “American Mathematical Monthly” 1975, 82, pp. 701-730.

J.F. BANZHAF, *Weighted Voting doesn't Work: a Mathematical Analysis*, "Rutgers Law Review" 1965, 19, pp. 317-43.

C. BERTINI, G. GAMBARELLI, I. STACH, *Apportionment Strategies for the European Parliament*, [in:] G. GAMBARELLI,, M. HOLLER, eds., *Power Measures III* , Special Issue of "Homo Oeconomicus", 22 (4) München: Accedo, 2005, pp. 589-604.

G. GAMBARELLI, *Minimax Apportionments*, "Group Decision and Negotiation" 1999, 8, pp. 441-461.

G. GAMBARELLI, J. HOLUBIEC, *Power Indices and Democratic Apportionment*, [in:] M. FEDRIZZI, J. KACPRZYK, eds., *Proceedings of the 8-th Italian-Polish Symposium on Systems Analysis and Decision Support in Economics and Technology*, Warsaw: Onnitech Press, 1990, pp. 240-255.

G. GAMBARELLI, G. OWEN, *The coming of Game Theory*, [in:] G. GAMBARELLI,, ed., *Essays on Cooperative Games - in honor of Guillermo Owen*, Special Issue of "Theory and Decision", Vol. 36, Dordrecht: Kluwer Academic Publishers, 2004, pp. 1-18.

G. GAMBARELLI, A. PALESTINI, *Minimax Multi-District Apportionments*, [in:] G. GAMBARELLI, ed., *Power Measures IV*, Special Issue of "Homo Oeconomicus", Vol. 23, München: Accedo Verlag, 2007, pp. 335-356.

D. SCHMEIDLER, *The nucleolus of a characteristic function game*, "SIAM Journal of Applied Mathematics" 1969, 17, pp. 1163-1170.